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**Optimality of deductible for Yaari's model: a reappraisal**

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# Optimality of deductible for Yaari's model: a reappraisal

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## Abstract

The main purpose of this paper is to show that left monotone risk aversion, a meaningful refinement of strong risk aversion, characterizes Yaari's decision makers for whom deductible insurance is optimal. A second goal is to offer a detailed proof of the deductible's computation, which proves the tractability of Yaari's model under left-monotone risk aversion.

*Keywords:* Yaari's model, Jewitt's left-monotone risk aversion, optimality of deductible.

*JEL classification:* D80, D81.

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## 1. Introduction

In the framework of EU model, Arrow (1965) proved that for a given premium, the optimal insurance contract for a EU risk averse decision maker is a contract with deductible. Gollier and Schlesinger (1996) obtained a nice generalization of this result by proving that this result holds also under strong aversion, whatever be the decision maker's decision model under risk.

Vergnaud (1997) refined this result by proving that for any left monotone risk averse decision maker (not necessarily strongly risk averse), whatever be the decision model under risk, the optimal contract for a given premium is a deductible policy.

This last result is important since strong risk aversion is disputable in some situations, while Jewitt (1989)'s refinement i.e. left monotone risk aversion appears to be better adapted to insurance. This adds further justification to RDEU (rank-dependent expected utility) models and in particular to Yaari (1987)'s model that allow the decision maker to be left monotone

risk averse without being strongly risk averse, which is impossible in the EU model, see Chateauneuf et al. (2004).

The goal of the present paper is to revisit the optimality of deductible in the framework of Yaari's model. Actually we show in section 4 that while left-monotone risk aversion is sufficient for a Yaari's decision maker to prefer deductible (a known result since Vergnaud (1997), for which we propose what we hope to be a more direct proof, actually as *Vergnaud* whatever be the decision maker's decision model), it turns out that for Yaari's decision maker left-monotone risk aversion is also a necessary condition for optimality of deductible.

In actual fact a main new result of this paper is to prove that optimality of deductible characterizes left-monotone risk averse Yaari's decision makers.

Moreover it is clear that once the optimality of a deductible policy for the insurer has been established, the question of computing her optimal level of deductible has to be settled.

As pointed out in Chateauneuf et al. (1997), Doherty and Eeckhoudt (1995) have shown that this question is very tractable in Yaari's model, when dealing with strongly risk averse insurers. It turns out that Chateauneuf et al. (1997) stated a theorem proving that the computation of the deductible remains very tractable for Yaari's decision maker merely assumed to be left monotone risk averse, but in fact the proof of this theorem has never been published, so a second goal of this paper is to fill this gap, see Section 5.

The paper is organized as follows: section 2 describes the framework and gives the main definitions, section 3 introduces left monotone increase in risk, sections 4 and 5 are devoted to our main results respectively characterization of left monotone risk averse Yaari's decision maker through optimality of deductible and computation of the optimal level of deductible. Finally, section 6 concludes the paper.

## 2. Framework and Definitions

In this section, we describe the decision maker's preference and the structure of insurance contract. We assume the decision maker chooses his preference through Yaari's model, a particular case of rank-dependent expected utility.

### 2.1. Yaari's Model

Under the rank-dependent expected utility (RDEU) model a decision maker is characterized by a utility function  $u$  and an increasing probability-transformation function  $f : [0, 1] \rightarrow [0, 1]$  that satisfies  $f(0) = 0$ ,  $f(1) = 1$ . Such a decision maker prefers a random variable  $X$  to a random variable  $Y$  if and only if  $V(X) > V(Y)$  where the functional  $V$  is given by

$$\begin{aligned} V(Z) = V_{u,f}(Z) &= - \int_{-\infty}^{\infty} u(x) df(P(Z > x)) = - \int_{-\infty}^{\infty} u(x) df(1 - F(x)) \\ &= \int_{-\infty}^0 [f(P(u(Z) > t)) - 1] + \int_0^{\infty} f(P(u(Z) > t)) dt, \end{aligned}$$

see Quiggin (1982); Chateauneuf et al. (1997). The Yaari functional is the special case of  $V(Z)$  where  $V(Z) = V_{I,f}(Z)$ . In fact the utility under certainty is the identity function  $u(x) \equiv x$ , see Yaari (1987).

In the context of insurance, we address prospects of the form  $W - D$ , such that  $W$  is a wealth endowment and  $D$  is a risky but insurable damage (defined on the support  $[0, W]$ ). Therefore, the Yaari functional  $V$  is defined in terms of the damage distribution  $F(D)$  as follows:

$$V = \int_{-\infty}^{+\infty} (W - D) df(F(D)). \quad (1)$$

### 2.2. Insurance contracts with Deductible structure

Consider a decision maker with initial deterministic wealth  $W > 0$  and possible random damage  $D \geq 0$  with distribution function  $F$  buying an insurance i.e. an indemnity  $I(D)$  such that:

1.  $0 \leq I(D) \leq D$ ,  $\forall D \in \mathbb{R}^+$
2. It has a price  $\pi$  given by  $\pi = (1 + m)E(I(D))$  where  $m \geq 0$  is the loading factor.

The decision maker will be said to express preference for deductible if for a given premium  $\pi$  among all the possible indemnities  $I$  satisfying (1) and (2), he prefers the indemnity  $I_d(D)$ ,

$$I_d(D) = (D - d)_+$$

where  $d$  is the level of the deductible.

In such a contract, the future wealth  $W_d$  of the decision maker is  $W_d = W - \pi - d + (D - d)_+$ . Therefore, the utility  $u(d)$  of this contract is given under Yaari's model by:

$$u(d) = W - (1 + m) \int_d^{+\infty} (1 - F(t))dt - d + \int_0^d f(F(t))dt. \quad (2)$$

### 3. Left monotone increase in risk

In the framework of expected utility (EU), a random variable  $Y$  is a mean preserving spread (MPS) of a random variable  $X$  if and only if all risk averse expected utility maximizers prefer  $X$  to  $Y$ . It should be noted that even in EU, there are some counter-intuitive examples. For instance, if a risk averse expected utility maximizer  $D_1$  is ready to pay  $c$  to exchange  $Y$  for a less risky  $X$  (i.e.  $Y$  MPS  $X$ ), and if an expected utility maximizer  $D_2$  is more risk averse than  $D_1$ , it can happen that  $D_2$  is ready to pay only  $c' < c$  for the same exchange. This single example proved that this notion of increasing risk, MPS, is not universal.

The left monotone order have been constructed initially by Jewitt, see Jewitt (1989), to solve the problem arised in insurance by MPS. This notion of increasing risk seems to be linked with the EU model but in fact there is a model-free equivalent definition of left monotone increase in risk.

#### *Definition 1*

For random variables  $X$  and  $Y$  with the same mean,  $Y$  is a left monotone increase in risk of  $X$  if  $\int_{-\infty}^{F_Y^{-1}(p)} F_Y(p) \geq \int_{-\infty}^{F_X^{-1}(p)} F_X(p)$ ,  $\forall p \in [0, 1]$ . Let us recall that for any distribution  $F$  i.e. any mapping  $F : \mathbb{R} \rightarrow \mathbb{R}$  non-decreasing, right-continuous such that  $\lim_{t \rightarrow -\infty} F(t) = 0$ ,  $\lim_{t \rightarrow +\infty} F(t) = 1$ ,  $F^{-1} : [0, 1] \rightarrow \mathbb{R}$  is defined  $\forall p \in [0, 1]$  by  $F^{-1}(p) = \inf \{t \in \mathbb{R}, F(t) \geq p\}$ . Note that  $F^{-1}(0) = -\infty$ .

In this definition, the upper limits of integration are arbitrary quantiles corresponding to equal probability level  $p$ . In fact,  $Y$  is a left monotone increase in risk of  $X$  if  $Y$  has more weight in the lower tail than  $X$ .

Chateauneuf et al. (2004) have been proved that when  $X$  and  $Y$  are discrete, with the same mean, left monotone increase in risk can be obtained by a finite sequence of following corresponding Pigou-Dalton transfers. To elaborate such a transformation, we consider the following generating process.

Let  $X$  and  $Y$  be two discrete random variables with distributions  $\mathcal{L}(X) = (x_1, p_1; x_2, p_2; x_3, p_3; x_4, p_4)$  where  $x_1 < x_2 < x_3 < x_4$  and

$$\mathcal{L}(Y) = (x_1 - \epsilon p_3, p_1; x_2, p_2; x_3 + \epsilon p_1, p_3; x_4, p_4), \quad (3)$$

where the outcomes are again in non-decreasing order.

One can prove that  $E(X) = E(Y)$  and  $Y$  is a left monotone increase in risk of  $X$ . In fact for any  $X$  and  $Y$  such that  $E(X) = E(Y)$ ,  $Y$  is a left monotone increase in risk of  $X$  if and only if  $Y$  can be obtained from  $X$  to a finite sequence of Pigou-Dalton transfers as above. In this spread, the minimal outcome is always spread out, but not necessarily the maximal outcome.

*Lemma 1*

For every pair  $(X, Y)$  of discrete random variables with  $E(X) = E(Y)$  such that  $Y$  is a left monotone increase in risk of  $X$ ,  $Y$  can be reached from  $X$  by a finite sequence of transfers as in (3).

**Proof:** See Chateauneuf et al. (2004). □

The following Definition 2 and Lemma 2 taken from Landsberger and Meilijson (1994) will be of great help for some proofs.

*Definition 2*

Distribution  $G$  is a left-monotone simple spread of  $F$  if

1.  $E(G) = E(F)$
2.  $\exists p_0 \in (0, 1)$  such that:
 

$p \leq p_0 \implies \begin{aligned} (2.1) \quad & G^{-1}(p) \leq F^{-1}(p) \\ (2.2) \quad & d(p) = F^{-1}(p) - G^{-1}(p) \text{ is non-increasing on } (0, p_0] \end{aligned}$

$p > p_0 \implies (2.3) \quad G^{-1}(p) \geq F^{-1}(p).$

*Lemma 2*

If  $G$  is a left-monotone simple spread of  $F$  then  $F$  is left-monotone less risky than  $G$ .

### 3.1. Left monotone risk aversion

A decision maker is left monotone risk averse if and only if for every  $X$  and  $Y$  such that  $Y$  is a left monotone increase in risk of  $X$  then he prefers  $X$  to  $Y$ . The left monotone risk aversion is a weaker notion of risk aversion compare to the mean preserving spreads (strong risk aversion) but stronger than the one based on the preference for the expected value of a random variable to the random variable itself. Next we discuss the left monotone increasing in risk in the Yaari's framework.

#### Lemma 3

Any Yaari decision maker is a left monotone increase in risk if and only if the probability transformation function is star shaped<sup>1</sup> at 1 i.e.  $\frac{1-f(p)}{1-p}$  is an increasing function of  $p$  on  $[0, 1)$ .

**Proof:** See Chateauneuf et al. (1997). □

## 4. Optimality of deductible characterizes left monotone risk averse Yaari's decision maker

We first show that preference for deductible implies left-monotone risk aversion.

#### Theorem 1

Any Yaari's decision maker who has preference for deductibles with any given premium is a left-monotone risk averse.

**Proof:** As we have already mentioned, Chateauneuf et al. (2004) have been shown that a decision maker is left-monotone risk averse if and only if the decision maker prefers any discrete random variable  $X$  such that  $\mathcal{L}(X) = (x_1, p_1; x_2, p_2; x_3, p_3; x_4, p_4)$  to any random variable  $Y$  such that  $\mathcal{L}(Y) = (x_1 - \epsilon p_3, p_1; x_2, p_2; x_3 + \epsilon p_1, p_3; x_4, p_4)$  where  $\epsilon \geq 0$ . Therefore in order to prove that a decision maker who has preference for deductibles turns out to be a left-monotone decision maker, it is sufficient to prove the following Lemma.

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<sup>1</sup>A function  $f \in F$  is star-shaped at  $m$ , if:

$$\frac{f(m)-f(p)}{m-p}$$

is an increasing function of  $p$  on  $[0, m) \cup (m, 1]$ .



*Lemma 4*

Any decision maker who exhibits preference for deductible will prefer  $\mathcal{L}(X) = (x_1, p_1; x_2, p_2; x_3, p_3; x_4, p_4)$  to  $\mathcal{L}(Y) = (x_1 - \epsilon p_3, p_1; x_2, p_2; x_3 + \epsilon p_1, p_3; x_4, p_4)$ , [Recall that through the definitions of the “ $\mathcal{L}$ ”, one has  $p_i \geq 0$ ,  $\sum_{i=1}^4 p_i = 1$  and  $x_1 < x_2 < x_3 < x_4$  and  $x_1 - \epsilon p_3 < x_2 < x_3 + \epsilon p_1 < x_4$ ].

**Proof:** Indeed it is enough to show that  $\exists w > 0, D \geq 0, m \geq 0$  and  $d \geq 0$  such that:

$$X = W - D - \pi + I_d(D)$$

$$Y = W - D - \pi + I(D)$$

Let  $\mathcal{L}(D) = (0, p_4; x_4 - x_3, p_3; x_4 - x_2, p_2; d_1, p_1)$ ,  $d = x_4 - x_1$  and  $d_1$  be such that  $d_1 - d - \epsilon p_3 > 0$ .

Note that  $d_1$  is consistent since  $d_1 > x_4 - x_2$ , actually  $d_1 > x_4 - x_1 + \epsilon p_3$ . Note also that  $d_2 = x_4 - x_2 < d = x_4 - x_1 < d_1$ . Therefore  $E((D - d)_+) = p_1(d_1 - d)$  and  $\pi$  is necessarily of the type  $\pi = (1 + m)p_1(d_1 - d)$  for some  $m > 0$ .

We need now to see if there exists some  $W$  such that our “new”  $X = W - D - \pi + I_d(D)$  actually satisfies:

$$\mathcal{L}(X) = (x_1, p_1; x_2, p_2; x_3, p_3; x_4, p_4)$$

Considering the four states  $s_i$  related to  $p_i$  we must have:

- (i)  $x_1 = W - d_1 - \pi + d_1 - d$
- (ii)  $x_2 = W - (x_4 - x_2) - \pi + 0$
- (iii)  $x_3 = W - (x_4 - x_3) - \pi + 0$
- (iv)  $x_4 = W - 0 - \pi + 0$

It is immediate that  $W = \pi + x_4$  is convenient. So for such a choice of  $W, D, \pi, d$  and  $m$  we actually get that our initial  $X$  satisfies  $X = W - D - \pi + I_d(D)$ .

It remains to check if one has actually that our initial  $Y$  is equal to  $W - D - \pi + I(D)$  where  $I(D)$  is a convenient indemnity idem est satisfying (i) and (ii).

Since (iii):  $I(D) = Y + D + \pi - W$ , one gets  $E(I(D)) = E(Y) + E(D) + \pi - W$  but  $E(Y) = p_1 x_1 + p_2 x_2 + p_3 y_3 + p_4 x_4 = E(X)$  so  $E(I(D)) = E(X) + E(D) + \pi - W = E(I_d(D))$ , therefore (ii) is satisfied.

Let us come to (i), from (iii) we get denoting  $d_i$  the damage if state  $s_i$  (related to probability  $p_i$ ) occurs:  $I(d_1) = x_1 - \epsilon p_3 + d_1 + \pi - x_4 - \pi = d_1 - (x_4 - x_1 + \epsilon p_3)$ .

Note that since  $d_1$  has been chosen such that  $d_1 > (x_4 - x_1 + \epsilon p_3)$  we actually have  $0 \leq I(d_1) \leq d_1$ ,  $I(d_2) = x_2 + x_4 - x_2 - x_4$  so  $0 \leq I(d_2) \leq d_2$  and  $I(d_3) = x_3 + \epsilon p_1 + x_4 - x_3 - x_4$  i.e.  $I(d_3) = \epsilon p_1$  but  $x_3 + \epsilon p_1 < x_4$  hence  $0 \leq I(d_3) \leq d_3$ , also  $I(d_4) = x_4 + 0 - x_4$  hence  $0 \leq I(d_4) \leq d_4$  which complete the proof of Lemma 4.  $\square$

*Remark 1*

Note that if we had required that indemnities should satisfy the Moral Hazard requirement i.e. that what remains to be paid by the decision maker namely  $D - I(D)$  should increase with the amount of the damage our Lemma 4 would remain valid. Actually:  $d_4 - I(d_4) = 0 < d_3 - I(d_3) = x_4 - x_3 - \epsilon p_1 < d_2 - I(d_2) = x_4 - x_2 < d_1 - I(d_1) = x_4 - x_1 + \epsilon p_3$ .

*Remark 2*

The proof of Theorem 1 shows that it is enough that a Yaari's decision maker has preference for deductible only in case of finite discrete random losses, in order to be a left-monotone risk averter.

*Theorem 2 (Vergnaud (1997))*

Any left-monotone risk-averse decision maker has preference for deductible.

**Proof:** Consider a left-monotone decision maker with initial deterministic wealth  $W$  and possible stochastic loss  $D \geq 0$ , buying an insurance with indemnity  $I(\cdot)$  where  $0 \leq I(t) \leq t \forall t \in \mathbb{R}$  at price  $\pi = (1 + m)E(I(D))$ .

We intend to show that this decision maker will actually buy the insurance with deductible  $d$  where indeed  $E((D - d)_+) = \frac{\pi}{1 + m}$ .

From Lemma 2 it is enough to prove that  $Z$  is a left-monotone spread of  $Z_d$  where  $Z = W - \pi - D + I(D)$ ,  $Z_d = W - \pi - D + I_d(D)$  and  $I(D) = (D - d)_+$ .

Since for any random variable  $T$  and any  $a \in \mathbb{R}$  one has  $F_{T+a}^{-1} = F_T^{-1} + a$ , it is enough to prove that  $-Y = -D + I(D)$  is a left monotone spread of  $-Y_d = -D + (D - d)_+$ .

Let  $p_0 = F_{-Y}(-d)$  the proof will be completed if we show that  $p_0 \in (0, 1)$  and that:

$$\begin{aligned} \forall p \leq p_0 \quad & F_{-Y}^{-1}(p) \leq F_{-Y_d}^{-1}(p) \\ & \text{and } F_{-Y_d}^{-1}(p) - F_{-Y}^{-1}(p) \text{ is non-increasing on } (0, p_0] \\ \forall p > p_0 \quad & F_{-Y}^{-1}(p) \geq F_{-Y_d}^{-1}(p). \end{aligned}$$

Note that  $-Y_d = \text{Max}(-D, -d)$ , and that  $-Y = -D + I(D) \geq -D$  therefore  $F_{-Y}(t) \leq F_{-D}(t) \forall t \in \mathbb{R}$ . Hence

$$\begin{aligned} t < -d \quad & F_{-Y_d}(t) = 0 \leq F_{-Y}(t) \\ t \geq -d \quad & F_{-Y}(t) \leq F_{-D}(t) = F_{-Y_d}(t) \end{aligned} \quad (4)$$

It turns out that  $p_0 > 0$ , otherwise  $p_0 = 0$  implies  $F_{-Y_d}(t) \geq F_{-Y}(t) \forall t$  and  $E(-Y_d) = E(-Y)$  entails  $-Y_d = -Y$  a contradiction.

Similarly  $p_0 = 1$  is impossible otherwise one would have  $F_{-Y_d}(t) \leq F_{-Y}(t) \forall t$ , hence  $-Y_d = -Y$  a contradiction. So from the above single-crossing of  $F_{-Y_d}$  and  $F_{-Y}$  we obtain that  $\exists p_0 \in (0, 1)$  namely  $p_0 = F_{-Y}(-d)$  such that

$$\begin{aligned} \forall p \leq p_0 \quad & F_{-Y}^{-1}(p) \leq F_{-Y_d}^{-1}(p) \\ \forall p > p_0 \quad & F_{-Y}^{-1}(p) \geq F_{-Y_d}^{-1}(p). \end{aligned}$$

It remains to prove that  $F_{-Y_d}^{-1}(p) - F_{-Y}^{-1}(p)$  is non-increasing on  $(0, p_0]$ . Since  $F_{-Y}^{-1}$  is non-decreasing it is enough to see that  $F_{-Y_d}^{-1}(p) = -d \forall p \in (0, p_0]$ .

From (4) we have  $F_{-Y_d}(-d) \geq p_0$  but  $-Y_d = \text{Max}(-D, -d)$  implies  $F_{-Y_d}(t) = 0 \forall t < -d$  hence  $F_{-Y_d}^{-1}(p_0) = -d$ . Furthermore, if  $0 < p < p_0$  indeed  $F_{-Y_d}^{-1}(p) \leq -d$ , but since  $F_{-Y_d}(t) = 0 \forall t < -d$ , this implies finally that  $F_{-Y_d}^{-1}(p) = -d \forall 0 < p \leq p_0$  which completes the proof.  $\square$

## 5. Computing the optimal level of deductible for a left monotone Yaari decision maker

Once the optimality of a deductible contract for the decision maker has been established, the question of computing his optimal level of deductible has to be settled.

Consider a left monotone Yaari decision maker with an initial wealth  $w \in \mathbb{R}_{++}$ , facing an insurable risky loss  $L$  with distribution function  $F : F(L) = P(L \leq \ell), \forall \ell \in \mathbb{R}$ . Assume that  $[0, \bar{\ell}]$  is the support of the random loss  $L$  and that the decision maker buys an indemnity  $I_d$  at price  $\pi$  given by  $\pi(I_d) = (1 + m)E(I_d)$ .

*Theorem 3*

A strict left monotone risk averse Yaari decision maker will purchase full insurance if

$$(1 + m)(1 - F(0)) - (1 - f(F(0))) < 0 \quad (5)$$

Otherwise,  $\bar{d}$  is an optimal level of deductible if and only if it satisfies

$$(1 + m)(1 - F(\bar{d}_-)) - (1 - f(F(\bar{d}_-))) \geq 0 \geq (1 + m)(1 - F(\bar{d})) - (1 - f(F(\bar{d}))) \quad (6)$$

*Remark 3*

If  $F$  is continuous, indeed the inequality (6) in theorem 3 reduces to the following simple equation:

$$(1 + m)(1 - F(\bar{d})) - (1 - f(F(\bar{d}))) = 0$$

**Proof:** Let us consider the simplest case which is named case 1.

Case 1: The distribution function  $F$  of the loss  $L$  is assumed to be strictly increasing on  $[0, \bar{\ell}]$  and continuous on  $\mathbb{R}$ .

We know that the decision maker aims at maximizing over  $[0, \bar{\ell}]$ , the function:

$$u(d) = w - d - (1 + m) \int_d^\infty (1 - F(t))dt + \int_0^d f(F(t))dt$$

where  $u(\cdot)$  is well defined and continuous on  $[0, \bar{\ell}]$ , so  $\text{Max } u(d)$  over  $[0, \bar{\ell}]$  exists.

Furthermore from continuity of  $f$  and  $F$ , we have  $u'(d)$  exists on  $[0, \bar{\ell}]$  and  $u'(d) = (1 + m)(1 - F(d)) - (1 - f(F(d)))$ ,  $\forall d \in [0, \bar{\ell}]$ . Since  $F$  is strictly increasing and  $F(\bar{\ell}) = 1$  one has  $F(d) < 1 \forall d \in [0, \bar{\ell}]$ , hence  $u'(d) = (1 - F(d))((1 + m) - \frac{1-f(F(d))}{1-F(d)})$ ,  $\forall d \in [0, \bar{\ell}]$ .

Therefore on  $[0, \bar{\ell}]$  the sign of the derivative is equal to the sign of  $g(d) = (1 + m) - \frac{1-f(F(d))}{1-F(d)}$  i.e.  $\text{sign } u'(d) = \text{sign } g(d)$ .

Let us assume that the decision maker is a strict left-monotone risk averter so  $h(d) = \frac{1-f(F(d))}{1-F(d)}$  is strictly increasing on  $[0, \bar{\ell}]$  and consequently  $g(d)$  is strictly decreasing on  $[0, \bar{\ell}]$ , note that here  $F(0) = 0$  so  $g(0) = m$ . Therefore if  $m = 0$ ,  $u'(0) = 0$  and  $u'(d) < 0 \forall d \in [0, \bar{\ell}]$ .

Since  $u$  is continuous on  $[0, \bar{\ell}]$  this implies that  $\text{Max } u(d)$  over  $[0, \bar{\ell}]$  is obtained at the unique point  $d = 0$ . That is if  $m = 0$ , the decision maker buys full insurance.

Assume now that  $m > 0$  therefore  $u'(0) > 0$ , either  $g(d) > 0 \forall d \in [0, \bar{\ell}]$  i.e.  $\lim_{p \rightarrow 1} \frac{1-f(p)}{1-p} \leq 1 + m$  and  $\text{Max } u(d)$  over  $[0, \bar{\ell}]$  will be obtained for  $d = \bar{\ell}$ , so the decision maker will not buy insurance, or  $\lim_{p \rightarrow 1} \frac{1-f(p)}{1-p} > 1 + m$ , and therefore  $u'$  will be first positive and then negative and by continuity of  $u'$  on  $[0, \bar{\ell}]$ , there exist a unique  $\bar{d} \in (0, \bar{\ell})$  where  $u'(\bar{d}) = 0$  and at that point one gets the optimal deductible of the decision maker.

### Summary of the results in case 1

If  $\underline{m} = 0$  i.e. fair insurance, the decision maker will buy full insurance.

If  $\underline{m} > 0$  denoting  $h(1_-) = \lim_{p \rightarrow 1} \frac{1-f(p)}{1-p}$  (note that this limit exists and is finite since  $\frac{1-f(p)}{1-p}$  is increasing on  $[0, 1]$ ). Either  $h(1_-) \leq 1 + m$  and the decision maker will not buy insurance or  $h(1_-) > 1 + m$  and the decision maker will choose the level of deductible  $\bar{d}$  which is the unique solution of  $u'(\bar{d}) = 0$  i.e.  $\bar{d}$  such that  $(1 + m)(1 - F(\bar{d})) = 1 - f(F(\bar{d}))$ .  $\square$

Case 2: We now switch to the general case which needs the preliminary lemmas 5 and 6.

#### *Lemma 5*

Let  $u : [a, b] \rightarrow \mathbb{R}$  be continuous and such that  $u'_+(\cdot)$  exists on  $(a, b)$  with  $u'_+(x) \leq 0 \forall x \in (a, b)$  then  $u$  is non-increasing on  $[a, b]$ .

**Proof:** Let us first prove that  $\forall x \in (a, b)$  and  $y \in (x, b]$  one has  $u(y) \leq u(x)$ . So take  $x \in (a, b)$ . By hypothesis  $u'_+(x) \leq 0$ .

• Take  $\epsilon > 0$  arbitrary, from  $u'_+(x) \leq 0$ , it comes that there exists  $y_0 \in (x, b]$  such that  $\forall y \in (x, y_0]$  one has:

$$u(y) - u(x) \leq \epsilon(y - x)$$

or else  $f(y) = u(y) - \epsilon y \leq u(x) - \epsilon x = f(x)$ .

• Let us prove that in fact  $f(y) \leq f(x) \forall y \in (x, b]$ . Let us define  $E$ :  
 $E = \{z \in (x, b] \text{ s.t. } y \in (x, z] \Rightarrow f(y) \leq f(x)\}$ ,  $E \neq \emptyset$  since  $y_0 \in E$ .  $E$  is bounded above by  $b$ , so  $\text{Sup}E$  exists. Denote  $M =: \text{Sup}E$ .

Let us prove that  $M \in E$ . Actually by definition of  $M$ , for any  $z \in (x, M)$  one has  $f(z) \leq f(x)$ , take  $z_n \in (x, M)$  with  $z_n \uparrow M$  one has  $f(z_n) \leq f(x) \forall n$ , since  $f$  is continuous one gets  $f(M) = \lim f(z_n) \leq f(x)$ , so  $M \in E$ . The

proof will be completed if we show that  $M = b$ .

• Assume  $M < b$  and let us show a contradiction. Since  $u'_+(M) \leq 0$ ,  $\exists y_1(\epsilon) > M$  where  $y_1(\epsilon) \in (M, b]$  such that  $u(y) - u(M) \leq \epsilon(y - M) \forall y \in (M, y_1]$  hence  $f(y) \leq f(M)$ , and therefore since  $f(M) \leq f(x)$  one gets  $f(y) \leq f(x) \forall y \in (x, y_1]$  a contradiction since  $y_1 > \text{Sup}E$ .

Therefore  $\forall \epsilon > 0$  one has  $u(y) - \epsilon y \leq u(x) - \epsilon x \forall y \in (x, b]$  so  $u(y) \leq u(x) \forall y \in (x, b]$ , i.e.  $\forall x \in (a, b)$  and  $y \in (x, b]$  one has  $u(y) \leq u(x)$ . Remains to show  $u(x) \leq u(a) \forall x \in [a, b]$ . But let  $x \in (a, b]$  and take  $a < x_n < x$  one has  $u(x) \leq u(x_n)$  let  $x_n \downarrow a$ , but  $u$  continuous implies  $u(a) = \lim u(x_n) \geq u(x)$  which completes the proof.  $\square$

*Lemma 6*

Let  $u : [a, b] \rightarrow \mathbb{R}$  be continuous and such that  $u'_+(\cdot)$  exists and strictly negative on  $J = (a, b)$  where  $J \neq \emptyset$  then  $u$  is strictly decreasing on  $[a, b]$ .

**Proof:** Let us first prove that for any given  $x \in (a, b)$  one has  $y \in (a, b]$   $y > x$  implies  $u(y) < u(x)$ . From  $u'_+(x) < 0$  i.e.  $\lim_{h \rightarrow 0^+} \frac{u(x+h) - u(x)}{h} < 0$  where  $x+h \in (x, b]$ , it comes that there exists  $y_0 \in (x, b]$  such that  $u(y) < u(x) \forall y \in (x, y_0]$ . If  $y_0 = b$  the proof is completed. If  $y_0 < b$  it is enough to show that  $u$  is non-increasing on  $(x, b]$  since  $y \in (x, b]$   $y > y_0$  will imply  $u(y) \leq u(y_0)$  but  $u(y_0) < u(x)$  which implies  $u(y) < u(x)$ .

Let  $E = \{z \in (x, b] \text{ s.t. } u(t) \leq u(x) \forall t \in (x, z]\}$ . One has  $E \neq \emptyset$  since  $y_0 \in E$ .  $E$  is bounded above by  $b$ , so  $\text{Sup}E$  exists. Denote  $M =: \text{Sup}E$ .

Let us prove that  $M \in E$ . Actually by definition of  $M$ , for any  $z \in (x, M)$  one has  $u(z) \leq u(x)$ , take  $z_n \in (x, M)$  with  $z_n \uparrow M$  one has  $u(z_n) \leq u(x) \forall n$ , since  $u$  is continuous one gets  $u(M) = \lim u(z_n) \leq u(x)$ , so  $M \in E$ .

Suppose  $M < b$ , then from  $u'_+(M) < 0$  there exists  $z_1 \in (M, b]$  such that  $u(t) < u(M) \forall t \in (M, z_1]$ , so since  $u(M) \leq u(x)$ , one gets  $u(t) \leq u(x) \forall t \in (x, z_1]$  and  $z_1 > M$  contradicts the definition of  $M$ , so  $M = b$ .

It remains to prove that  $u(a) > u(x) \forall x \in (a, b]$ . Let  $a < x_n < x$ ,  $x_n \downarrow a$ , one has  $u(x) < u(x_n) \forall n$ , so  $u(x) < u(x_{n_0}) \leq u(x_n) \ n \geq n_0$ ,  $u(x_n) \uparrow u(a)$  since  $u$  is continuous so  $u(x) < u(a)$  which completes the proof.  $\square$

Theorem 3 is proved based on the results of lemma 5 and lemma 6 as follows.

**Proof of Theorem 3:** Let  $[0, \bar{\ell}]$  be the support of the random variable of losses  $L$  where the c.d.f is  $F$  so  $\bar{\ell} = \inf \{\ell \geq 0, F(\ell) = 1\}$ . One has:

$$u'_+(d) = (1+m)(1-F(d)) - (1-f(F(d))) \text{ on } [0, \bar{\ell})$$

$$u'_-(d) = (1+m)(1-F(d_-)) - (1-f(F(d_-))) \text{ on } (0, \bar{\ell}]$$

By hypothesis  $F(\ell) < 1 \forall \ell \in [0, \bar{\ell}]$ ,  $f$  is strictly increasing and continuous on  $[0, 1]$  with  $f(0) = 0$ ,  $f(1) = 1$  and  $\frac{1-f(p)}{1-p}$  is non-decreasing on  $[0, 1)$ .

Therefore:

$$u'_+(d) = g(d)(1 - F(d)) \quad \forall d \in (0, \bar{\ell})$$

$$u'_-(d) = h(d)(1 - F(d_-)) \quad \forall d \in (0, \bar{\ell})$$

where  $g(d) = 1 + m - \frac{1-f(F(d))}{1-F(d)}$ ,  $h(d) = g(d_-)$ .

Note that  $h(d) \geq g(d) \forall d \in (0, \bar{\ell})$ , hence  $h(d) \leq 0$  implies  $g(d) \leq 0$  and  $g(d) \geq 0$  implies  $h(d) \geq 0$ .

So on  $(0, \bar{\ell})$ ,  $u'_+(d) \leq 0 \Leftrightarrow g(d) \leq 0$  since  $1 - F(d) \geq 0$  and  $u'_-(d) \geq 0 \Leftrightarrow h(d) \geq 0$  since  $1 - F(d_-) \geq 0$  as well.

Note that  $u'_+(0)$  exist and  $u'_+(0) = (1 - F(0))g(0)$ .

Note that  $g$  is defined on  $[0, \bar{\ell}]$  and is non-increasing.

Case 1: (1)  $1 + m < \frac{1-f(F(0))}{1-F(0)}$

Let us show that in such a case  $\text{Max } u(d)$  over  $[0, \bar{\ell}]$  (which exists since  $u$  is continuous) is obtained for  $\bar{d} = 0$ . So in case 1  $\bar{d} = 0$  i.e. there exists a unique optimal deductible which proves to be full insurance. Note that this is (5) of theorem 3.

**Proof:** By hypothesis  $g(0) < 0$  but  $g$  is non-increasing on  $[0, \bar{\ell}]$ , hence  $g(d) < 0 \forall d \in (0, \bar{\ell})$  so  $u'_+(d) < 0 \forall d \in (0, \bar{\ell})$ . Since  $u$  is continuous on  $[0, \bar{\ell}]$ , from Lemma 6  $u$  is strictly decreasing on  $[0, \bar{\ell}]$  hence the maximum of  $u$  on  $[0, \bar{\ell}]$  is uniquely obtained for  $d = 0$ , so  $\bar{d} = 0$  is the optimal deductible.  $\square$

Case 2: (2)  $\lim_{d \uparrow \bar{\ell}} \frac{1-f(F(d))}{1-F(d)} < 1 + m^1$

In such a case there exists a unique optimal deductible which is  $\bar{d} = \bar{\ell}$ , so the decision maker will ask for no insurance, and it is straightforward that (6) is satisfied with  $\bar{d} = \bar{\ell}$ .

**Proof:** From case 2,  $h(\bar{\ell})$  exists finite strictly positive, and since  $h$  is non-increasing on  $(0, \bar{\ell}]$ , one gets  $h(d) > 0$  on  $(0, \bar{\ell})$ , hence  $u'_-(d) > 0$  on  $(0, \bar{\ell})$ .

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<sup>1</sup>Note that  $\forall p \in [0, 1)$   $\frac{1-f(p)}{1-p} \geq \frac{1-0}{1-0} = 1$ , that  $\frac{1-f(p)}{1-p}$  is non-decreasing on  $[0, 1)$  so this means that here  $\lim_{p \uparrow 1} \frac{1-f(p)}{1-p}$  exists and is finite.

Since  $u$  is continuous on  $[0, \bar{\ell}]$ , it turns out from *Lemma 6'* (the version of *Lemma 6* where  $u'_-(\cdot)$  exists on  $(a, b)$  and  $u'_-(x) > 0 \forall x \in (a, b)$ ) that  $u$  is strictly increasing on  $[0, \bar{\ell}]$  and therefore that  $\text{Max } u(d)$  over  $[0, \bar{\ell}]$  is uniquely obtained at  $\bar{d} = \bar{\ell}$ .  $\square$

Case 3: Note that  $r(d) = \frac{1-f(F(d))}{1-F(d)}$  is well-defined  $\forall d \in [0, \bar{\ell}]$  (indeed we eliminate the case when  $F(0) = 1$ , since in this case the decision maker would suffer for no loss, hence would not like to insure) and positive, hence since  $r(d)$  is non-decreasing with  $d$ , we obtain that  $\lim_{d \uparrow \bar{\ell}} r(d)$  exists eventually equal to  $+\infty$ .

This last case 3 occurs when case 1 and case 2 are falsified, i.e. when :

$$(3) \quad \frac{1-f(F(0))}{1-F(0)} \leq 1 + m \leq \lim_{d \uparrow \bar{\ell}} \frac{1-f(F(d))}{1-F(d)}$$

Let us now consider  $E = \left\{ d \in [0, \bar{\ell}], \frac{1-f(F(d))}{1-F(d)} > 1 + m \right\}$

• Either  $E \neq \emptyset$  Since  $E$  is bounded from below,  $\text{Inf}(E)$  exists, let  $d_0 := \text{Inf}(E)$ .

From the definition of  $d_0$ , one has  $\frac{1-f(F(d))}{1-F(d)} > 1 + m \forall d \in [0, \bar{\ell}]$  such that  $d > d_0$ , since  $r(\cdot)$  is right-continuous on  $[0, \bar{\ell}]$  one gets  $\frac{1-f(F(d_0))}{1-F(d_0)} \geq 1 + m$ .

If  $d_0 = 0$  one has  $\frac{1-f(F(0))}{1-F(0)} = 1 + m$  and  $\frac{1-f(F(d))}{1-F(d)} > 1 + m \forall d \in (0, \bar{\ell})$ , hence  $g(d) < 0$  on  $(0, \bar{\ell})$ , so  $u$  is strictly decreasing on  $[0, \bar{\ell}]$ , and therefore there exists a unique optimal deductible for the decision maker i.e.  $\bar{d} = 0$ . To summarize the first sub-case is:

Case 3.1:

$$(3.1) \quad \frac{1-f(F(0))}{1-F(0)} = 1 + m < \frac{1-f(F(d))}{1-F(d)} \quad \forall d \in (0, \bar{\ell})$$

and then the unique optimal deductible for the decision maker is  $\bar{d} = 0$  i.e. the decision maker choose full insurance. Again (6) is satisfied with  $\bar{d} = 0$ .

Let us switch to the second case, where  $d_0 > 0$ . In such a case since  $d_0 = \text{Inf}(E)$ , one has  $\forall d \in [0, \bar{\ell}], d < d_0, \frac{1-f(F(d))}{1-F(d)} \leq 1 + m$ , hence  $\frac{1-f(F(d_0-))}{1-F(d_0-)} \leq 1 + m$ . Therefore the second sub-case is:

Case 3.2:  $\exists d_0 \in (0, \bar{\ell})$  such that:

$$(3.2) \quad \frac{1-f(F(d_0-))}{1-F(d_0-)} \leq 1 + m \leq \frac{1-f(F(d_0))}{1-F(d_0)}$$



In such a case  $\bar{d} = d_0$  is an optimal deductible. This optimal deductible is not unique if and only if there exists  $d'_0 > 0$ ,  $d'_0 < d_0$  such that:

$$\frac{1-f(F(d_-))}{1-F(d_-)} = \frac{1-f(F(d_{0-}))}{1-F(d_{0-})} = 1 + m \quad \forall d \in [d'_0, d_0]$$

in which case any  $\bar{d} \in [d'_0, d_0]$  is an optimal deductible. Note that (6) is satisfied for any  $\bar{d} \in [d'_0, d_0]$

**Proof:** For  $d \in (0, d_0)$ , we have  $h(d) \geq 0$ , therefore  $u'_-(d) \geq 0$ , hence Lemma 5' (the dual Lemma of Lemma 5) implies  $u$  is non-decreasing over  $[0, d_0]$ . Similarly, for  $d \in [d_0, \bar{\ell})$ , we have  $g(d) \leq 0$ , therefore,  $u'_+(d) \leq 0$ , hence Lemma 5 implies  $u$  is non-increasing over  $[d_0, \bar{\ell}]$ . Therefore  $d_0$  is an optimal deductible.

Since  $u(d)$  is non-decreasing over  $[0, d_0]$ , if the optimal deductible  $d_0$  is not unique, there exists  $d'_0 < d_0$  such that  $u(d) = u(d_0)$  for  $d \in [d'_0, d_0]$ , the interval is closed due to the continuity of  $u(\cdot)$ . Hence  $u'_-(d) = 0 \quad \forall d \in (d'_0, d_0)$  and we have  $h(d) = 0$  for  $d \in (d'_0, d_0]$ .

On the other hand if there exists  $d'_0 < d_0$  such that  $h(d) = 0$  for  $d \in (d'_0, d_0]$  then  $u'_-(d) \geq 0$  and  $u'_+(d) \leq 0$  for all  $d \in (d'_0, d_0]$ . Therefore,  $u$  is both non-decreasing and non-increasing over this interval. Hence  $u(d) = u(d_0)$  for  $d \in [d'_0, d_0]$ , in this case all the points  $d \in [d'_0, d_0]$  are optimal.  $\square$

We now need to consider the case:

- Or  $E = \emptyset$  In such a case we have:

$$(4) \quad \frac{1-f(F(d))}{1-F(d)} \leq 1 + m \quad \forall d \in [0, \bar{\ell}]$$

hence  $\lim_{d \uparrow \bar{\ell}} \frac{1-f(F(d))}{1-F(d)} \leq 1 + m$ , therefore from (3):

$$\text{Case 3.3: (3.3) } \lim_{d \uparrow \bar{\ell}} \frac{1-f(F(d))}{1-F(d)} = 1 + m$$

**Proof:** In this case, by the assumption, we have  $h(\bar{\ell}) = 0$ . Since  $h$  is non-increasing on  $(0, \bar{\ell}]$ , we can conclude that  $h(d) \geq 0$  on  $(0, \bar{\ell}]$ . Lemma 5' (the version of Lemma 5 where  $u'_-(\cdot)$  exists on  $(a, b)$  and  $u'_-(x) \geq 0 \quad \forall x \in (a, b)$ ) implies that  $u$  is non-decreasing on  $[0, \bar{\ell}]$ . Therefore,  $\bar{\ell}$  is an optimal deductible in this case.

Similar to the case 3.2, given that  $u(d)$  is non-decreasing and continuous,  $\bar{\ell}$  is not the unique optimal, if and only if there exists  $d'$  such that  $0 \leq d' < \bar{\ell}$  and  $\frac{1-f(F(d))}{1-F(d)} = 1 + m \quad \forall d \in [d', \bar{\ell})$  in which case any  $d \in [d', \bar{\ell}]$  is an optimal deductible. Again it is straightforward that (6) is satisfied  $\forall \bar{d} \in [d', \bar{\ell}]$ . This completes the proof of theorem 3.

Considering that  $F$  is continuous and strictly increasing and the Yaari decision maker is strongly risk averse, therefore Remark 3 results in the decision maker will not buy insurance when the loading factor is  $m \geq 1$  and  $m \geq 2$  respectively for  $f(p) = p^2$  and  $f(p) = p^3$ .

## 6. Conclusion

While it is known since Vergnaud (1997) that whatever be the decision maker's model under risk, left-monotone risk aversion, a meaningful refinement of strong risk aversion introduced by Jewitt (1989), implies optimality of deductible, the main purpose of this paper is to show that in fact within the Yaari's model left-monotone risk aversion does characterize the optimality of deductible insurance.

A second main goal of this paper is to show that for such left-monotone Yaari's risk averters, the computation of the deductible is very tractable. Chateauneuf et al. (1997) stated a theorem related to this point, but in fact the proof of this theorem has never been published, this paper aims to fill this gap.

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